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Hyper-Kähler quotients of solvable Lie groups

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Abstract

In this paper we apply the hyper-Kähler quotient construction to Lie groups with a left invariant hyper-Kähler structure under the action of a closed abelian subgroup by left multiplication. This is motivated by the fact that some known hyper-Kähler metrics can be recovered in this way by considering different Lie group structures on $\mathbb{H}^p \times \mathbb{H}^q$ ($\mathbb{H}$: the quaternions). We obtain new complete hyper-Kähler metrics on Euclidean spaces and give their local expressions.

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1. Introduction

Hyper-Kähler manifolds, which generalize the notion of Kähler manifolds, appear related to solutions of well-known equations in mathematical physics. A hyper-Kähler metric on a manifold M is a Riemannian metric g which is Kähler with respect to two anticommuting complex structures J_1 and J_2 on M.

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It is not easy to obtain explicit examples of such manifolds. Hyper-Kähler reduction [8] allows to construct hyper-Kähler manifolds from others admitting a group acting by tri-holomorphic isometries. Families of $4n$-dimensional hyper-Kähler quotients with a tri-holomorphic T^n-action were constructed in [5,2]. In particular, in [2] the geometry and topology of hyper-Kähler quotients of $\mathbb{H}^d$ by subtori of T^d has been studied.

The hyper-Kähler quotient construction has been also applied in [4] to the flat space $\mathbb{H}^d$ to obtain some monopole moduli space metrics in explicit form using [10,15], for instance the Taubian-Calabi [16] and the Lee–Weinberg–Yi metric [11]. These are constructed by considering the following actions of $\mathbb{R}$ on $\mathbb{H} \times \mathbb{H}^m$ (resp. $\mathbb{R}^m$ on $\mathbb{H}^m \times \mathbb{H}^m$):

$$\mathbb{R} \times \mathbb{H} \times \mathbb{H}^m \to \mathbb{H} \times \mathbb{H}^m, \quad (t, (q, w_1, \ldots, w_m)) \to (t+q, \mathrm{e}^{\mathrm{i}t} w_1, \ldots, \mathrm{e}^{\mathrm{i}t} w_m),$$
$$\mathbb{R}^m \times \mathbb{H}^m \times \mathbb{H}^m \to \mathbb{H}^m \times \mathbb{H}^m, \quad ((t_1, \ldots, t_m), (q_1, \ldots, q_m, w_1, \ldots, w_m))$$
$$\to (t_1 + q_1, \ldots, t_m + q_m, \mathrm{e}^{\mathrm{i}\langle\theta_1, T\rangle} w_1, \ldots, \mathrm{e}^{\mathrm{i}\langle\theta_m, T\rangle} w_m),$$

where $\theta \in \mathrm{GL}(m, \mathbb{R})$, $T = (t_1, \ldots, t_m)$, θ_β are the rows of θ and $\langle , \rangle$ is the Euclidean inner product in $\mathbb{R}^m$. The first action gives rise to the Taubian-Calabi metric, which coincides with the Taub-Nut metric for $q = 1$, and the second one corresponds to the Lee–Weinberg–Yi metric. We show that in both cases the metric can be recovered by endowing $\mathbb{H} \times \mathbb{H}^m$ (resp. $\mathbb{H}^m \times \mathbb{H}^m$) with a hyper-Kähler Lie group structure and taking the quotient with respect to a suitable closed abelian subgroup.

In the present work we study hyper-Kähler quotients starting from a Lie group G with a left invariant hyper-Kähler structure. Such a group is necessarily flat since it is Ricci flat and homogeneous (see [1]). It follows from [12] that G must be two-step solvable and when G is simply connected, G is a semidirect product of the form $\mathbb{H}^p \ltimes_\theta \mathbb{H}^q$, where θ is a homomorphism from $\mathbb{H}^p$ to T^q, a maximal torus in $\mathrm{Sp}(q)$ (see Proposition 3.1 and (13)). This leads us to get a characterization of hyper-Kähler Lie groups.

We take a connected closed abelian subgroup $\mathbb{R}^l$ ($l \leq p$) of $\mathbb{H}^p$ which acts on G by left translations, hence the action is free and the moment map has no critical points. This action is tri-Hamiltonian, therefore the hyper-Kähler quotient construction [8] can be applied. We prove that the metric obtained on the hyper-Kähler quotient is complete and the quotient is diffeomorphic to an Euclidean space. Since the $\mathbb{R}^l$-action commutes with an action of the torus T^q, if $l = p$ the $4q$-dimensional hyper-Kähler quotient admits a tri-holomorphic T^q-action. Such action has a unique fixed point when $p = q$. In this way we obtain new complete hyper-Kähler metrics which generalize the Taubian-Calabi and the Lee–Weinberg–Yi metrics. Using the same method as in [4,15,10], we obtain a local expression of the hyper-Kähler quotient metrics. This expression is given in terms of the structure constants of the corresponding Lie group $\mathbb{H}^p \ltimes_\theta \mathbb{H}^q$.

2. Preliminaries

Let $(\mathfrak{g}, g)$ be a metric Lie algebra, that is, $\mathfrak{g}$ is a Lie algebra endowed with an inner product g. The Levi–Civita connection associated to the metric can be computed by

$$2g(\nabla_X Y, Z) = g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y), \tag{1}$$

for any X, Y, Z in $\mathfrak{g}$.

A hypercomplex structure on $\mathfrak{g}$ is a triple of complex structures $\{J_\alpha\}_{\alpha=1,2,3}$ satisfying the quaternion relations

$$J_\alpha^2 = -\mathrm{id}, \quad \alpha = 1, 2, 3, \qquad J_1 J_2 = -J_2 J_1 = J_3,$$

together with the vanishing of the Nijenhuis tensor $N_\alpha(X, Y) = 0$, for any $X, Y \in \mathfrak{g}$ and $\alpha = 1, 2, 3$. Here, the Nijenhuis tensor stands for

$$N_\alpha(X, Y) = J_\alpha([X, Y] - [J_\alpha X, J_\alpha Y]) - ([J_\alpha X, Y] + [X, J_\alpha Y]),$$

where $X, Y \in \mathfrak{g}$.

Let $\mathfrak{g}$ be a Lie algebra endowed with a hypercomplex structure $\{J_\alpha\}_{\alpha=1,2,3}$ and an inner product g, compatible with the hypercomplex structure, that is

$$g(X, Y) = g(J_1 X, J_1 Y) = g(J_2 X, J_2 Y) = g(J_3 X, J_3 Y),$$

for all $X, Y \in \mathfrak{g}$. We will say that $(\mathfrak{g}, \{J_\alpha\}, g)$ is a hyper-Kähler Lie algebra when $(\mathfrak{g}, J_\alpha, g)$ is a Kähler Lie algebra, for each α, that is, $\nabla J_\alpha = 0$, where ∇ is the Levi–Civita connection of g. This is equivalent to $\mathrm{d}\omega_\alpha = 0$, where ω_α are the associated Kähler forms defined by $\omega_\alpha(X, Y) = g(J_\alpha X, Y)$, $X, Y \in \mathfrak{g}$.

If G is a Lie group with Lie algebra $\mathfrak{g}$ then the above structures on $\mathfrak{g}$ can be left translated to all of G obtaining invariant hyper-Kähler structures on G.

Note that a Lie group with an invariant hyper-Kähler structure is necessarily flat since a hyper-Kähler metric is Ricci flat and in the homogeneous case, Ricci flatness implies flatness (see [1]). Examples of non commutative Lie groups carrying a flat invariant metric are given by $T^k \ltimes \mathbb{R}^m$ where T^k is a torus in $\mathrm{SO}(m)$. The next proposition, which is a consequence of the characterization of flat Lie algebras given in [12], shows that this family of examples essentially exhausts the class (see also [3]). This will allow us to give a characterization of hyper-Kähler Lie algebras as a special class of subalgebras of $\mathbb{R}^s \times \mathfrak{e}(4q)$, where $\mathfrak{e}(4q) = \mathfrak{so}(4q) \ltimes \mathbb{R}^{4q}$ is the Euclidean Lie algebra.

Proposition 2.1 (Milnor [12]). *Let* $(\mathfrak{g}, g)$ *be a flat Lie algebra. Then* $\mathfrak{g}$ *decomposes orthogonally as*

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{h} \oplus \mathfrak{g}^1,$$

where $\mathfrak{z}(\mathfrak{g})$ *is the center of* $\mathfrak{g}$, $\mathfrak{h}$ *is an abelian Lie subalgebra, the commutator ideal* $\mathfrak{g}^1$ *is abelian and the following conditions are satisfied*:

(i) $\mathrm{ad} : \mathfrak{h} \to \mathfrak{so}(\mathfrak{g}^1)$ *is injective and* $\mathfrak{g}^1$ *is even dimensional*;
(ii) $\mathrm{ad}_X = \nabla_X$ *for any* $X \in \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{h}$.

In particular, $\mathfrak{g}$ *is isomorphic to a Lie subalgebra of* $\mathbb{R}^s \times \mathfrak{e}(\mathfrak{g}^1)$, *where* $s = \dim \mathfrak{z}(\mathfrak{g})$.

Proof. By [12] a flat Lie algebra $(\mathfrak{g}, g)$ decomposes orthogonally as

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{b}, \tag{2}$$

where $\mathfrak{h}$ is an abelian Lie subalgebra, $\mathfrak{b}$ is the abelian ideal defined by $\{B \in \mathfrak{g} : \nabla_B = 0\}$ and

$$\mathrm{ad}_X : \mathfrak{b} \to \mathfrak{b}$$

is skew-symmetric, for any $X \in \mathfrak{h}$. Note that the above conditions imply that ad_X is skew-symmetric on $\mathfrak{g}$ for any $X \in \mathfrak{h}$, hence,

$$\mathrm{ad}_X = \nabla_X, \text{ for any } X \in \mathfrak{h}. \tag{3}$$

The above equation and the choice of $\mathfrak{b}$ imply

$$\mathrm{ad} : \mathfrak{h} \to \mathfrak{so}(\mathfrak{g}) \tag{4}$$

is injective.

We notice next that the decomposition (2) implies that $\mathfrak{g}^1 \subseteq \mathfrak{b}$, hence $\mathfrak{b}$ decomposes orthogonally as

$$\mathfrak{b} = \mathfrak{v} \oplus \mathfrak{g}^1.$$

We show below that $\mathfrak{v} = \mathfrak{z}(\mathfrak{g})$, where $\mathfrak{z}(\mathfrak{g})$ denotes the center of $\mathfrak{g}$. In particular, $\mathfrak{g}$ will decompose orthogonally as

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{h} \oplus \mathfrak{g}^1,$$

with $\mathfrak{h}$ and $\mathfrak{g}^1$ abelian and such that (i) holds. To show that $\mathfrak{g}^1$ is even dimensional, assume that $\dim \mathfrak{g}^1 = 2m + 1$. Since ad_X, $X \in \mathfrak{h}$, is a commutative family of endomorphisms in $\mathfrak{so}(2m + 1)$, they are conjugate to elements in a maximal abelian subalgebra of $\mathfrak{so}(2m + 1)$, hence there exists $Z \in \mathfrak{g}^1$ such that $\mathrm{ad}_X(Z) = 0$ for any $X \in \mathfrak{h}$, therefore $Z \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}^1$, a contradiction.

Since $\mathrm{ad}_X : \mathfrak{g}^1 \to \mathfrak{g}^1$ is skew-symmetric, for any $X \in \mathfrak{h}$, then it preserves $\mathfrak{v}$. Therefore, $[X, \mathfrak{v}] \subset \mathfrak{v} \cap \mathfrak{g}^1 = 0$ for $X \in \mathfrak{h}$ and $\mathfrak{v} \subset \mathfrak{z}(\mathfrak{g})$ follows. On the other hand, if $Y \in \mathfrak{z}(\mathfrak{g})$, then:

$$0 = g([Y, X], U) = g(Y, [X, U]),$$

for every $X \in \mathfrak{h}$, $U \in \mathfrak{g}^1$, that is, $\mathfrak{z}(\mathfrak{g}) \perp \mathfrak{g}^1$ since $\mathfrak{g}^1 = [\mathfrak{h}, \mathfrak{g}^1]$. From $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{h} = 0$ one has that $\mathfrak{v} = \mathfrak{z}(\mathfrak{g})$.

Finally, using (1) one can compute $\nabla_Y = 0$ for $Y \in \mathfrak{z}(\mathfrak{g})$. This together with (3) implies (ii) and the proposition follows. □

We will say that two flat Lie algebras $(\mathfrak{g}_1, g_1)$ and $(\mathfrak{g}_2, g_2)$ are equivalent if there exists an orthogonal Lie algebra isomorphism $\eta : \mathfrak{g}_1 \to \mathfrak{g}_2$. Note that $\eta : \mathfrak{z}(\mathfrak{g}_1) \to \mathfrak{z}(\mathfrak{g}_2)$, $\eta : \mathfrak{g}_1^1 \to \mathfrak{g}_2^1$ and therefore $\eta : \mathfrak{h}_1 \to \mathfrak{h}_2$ (see Proposition 2.1). Let $\mathrm{ad}_i : \mathfrak{h}_i \to \mathfrak{so}(\mathfrak{g}_i^1)$, $i = 1, 2$, be the corresponding injective maps induced by the adjoint representation on $\mathfrak{g}_i$. Then the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{h}_1 & \xrightarrow{\mathrm{ad}_1} & \mathfrak{so}(\mathfrak{g}_1^1) \\ \eta \downarrow & & \downarrow I_\eta \\ \mathfrak{h}_2 & \xrightarrow{\mathrm{ad}_2} & \mathfrak{so}(\mathfrak{g}_2^1) \end{array}$$

where I_η denotes conjugation by η. It follows from Proposition 2.1 that every flat Lie algebra with $2m$-dimensional commutator and s-dimensional center is equivalent to $\mathbb{R}^s \times \mathbb{R}^k \ltimes_\rho \mathbb{R}^{2m}$, where $\rho : \mathbb{R}^k \to \mathfrak{so}(2m)$ is injective, $\rho(\mathbb{R}^k)\mathbb{R}^{2m} = \mathbb{R}^{2m}$, the only non zero Lie brackets being

$$[X, Y] = \rho(X)Y, \quad X \in \mathbb{R}^k,\ Y \in \mathbb{R}^{2m}.$$

Given a flat Lie algebra $\mathbb{R}^s \times (\mathbb{R}^k \ltimes_\rho \mathbb{R}^{2m})$, the family $\{\rho(T) : T \in \mathbb{R}^k\} \subseteq \mathfrak{so}(2m)$ is an abelian subalgebra, then it is conjugate by an element in SO($2m$) to a subalgebra of the following maximal abelian subalgebra of $\mathfrak{so}(2m)$:

$$\mathfrak{t}^m = \left\{ \begin{pmatrix} 0 & -\phi_1 & & & \\ \phi_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\phi_m \\ & & & \phi_m & 0 \end{pmatrix} : \phi_\alpha \in \mathbb{R} \right\}$$

with respect to an orthonormal basis $\{f_1, \ldots, f_{2m}\}$ of $\mathbb{R}^{2m}$. In particular, $k \leq m$ and we may assume that any flat Lie algebra is equivalent to a Lie algebra such that $\rho(\mathbb{R}^k) \subset \mathfrak{t}^m$.

Let $\theta = (\theta^\alpha_\beta)$ be the real $m \times k$ matrix of rank k such that

$$\rho(e_\alpha) = \begin{pmatrix} 0 & -\theta_1^\alpha & & & \\ \theta_1^\alpha & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\theta_m^\alpha \\ & & & \theta_m^\alpha & 0 \end{pmatrix}, \quad 1 \leq \alpha \leq k, \tag{5}$$

where $\{e_1, \ldots, e_k\}$ is an orthonormal basis of $\mathbb{R}^k$. The condition $\rho(\mathbb{R}^k)\mathbb{R}^{2m} = \mathbb{R}^{2m}$ is equivalent to the fact that every row θ_β of θ is non zero.

We introduce some notation that will be needed in the next result. Let $M(k, m; k)$ be the set of $m \times k$ real matrices of rank k. $M(k, m; k)$ can be embedded in End($\mathbb{R}^k$, $\mathfrak{so}(2m)$) by means of the inclusion ρ:

$$M(k,m;k) \hookrightarrow \mathrm{End}(\mathbb{R}^k, \mathfrak{so}(2m)), \quad \theta \mapsto \rho_\theta,$$

where ρ_θ is as in (5). We identify $M(k,m;k)$ with its image under ρ and let $\mathrm{O}(k)\times\mathrm{O}(2m)$ act on $M(k,m;k)$ as follows:

$$\mathrm{O}(k)\times\mathrm{O}(2m)\times M(k,m;k) \to M(k,m;k), \quad (A, B, \rho_\theta) \mapsto B\rho_{(\theta A)}B^{-1}, \tag{6}$$

where $B\rho_\theta B^{-1} \in \mathrm{End}(\mathbb{R}^k, \mathfrak{so}(2m))$ is defined by $B\rho_\theta B^{-1}(T) = B\rho_\theta(T)\,B^{-1}$, $T\in\mathbb{R}^k$. It follows from the definition of equivalence between flat Lie algebras that

$$\mathbb{R}^k \ltimes_{\rho_\theta} \mathbb{R}^{2m} \cong \mathbb{R}^k \ltimes_{\rho_{\theta'}} \mathbb{R}^{2m}$$

if and only if ρ_θ and $\rho_{\theta'}$ lie in the same $\mathrm{O}(k)\times\mathrm{O}(2m)$-orbit.

The next proposition summarizes the above results and gives the classification of flat Lie algebras that will be needed in the next section (see also [9]).

Proposition 2.2. *Let* $(\mathfrak{g}, g)$ *be a flat Lie algebra*, $\dim\mathfrak{g}^1 = 2m$, $\dim\mathfrak{z}(\mathfrak{g}) = s$. *Then there exists* $\theta = (\theta^\alpha_\beta) \in M(k,m;k)$ *such that* $\theta_\beta \neq 0$ *for every* $1 \le \beta \le m$ *and* $\mathfrak{g}$ *decomposes orthogonally as*

$$\mathfrak{g} \cong \mathbb{R}^s \times (\mathbb{R}^k \ltimes_{\rho_\theta} \mathbb{R}^{2m}),$$

where $\mathbb{R}^k \ltimes_{\rho_\theta} \mathbb{R}^{2m}$ *has an orthonormal basis* $\{e_1,\dots,e_k, f_1,\dots,f_{2m}\}$ *and* $T\in\mathbb{R}^k$ *acts on* $\mathbb{R}^{2m}$ *in the following way*:

$$\rho_\theta(T) = \begin{pmatrix} 0 & -\langle T,\theta_1\rangle & & & \\ \langle T,\theta_1\rangle & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\langle T,\theta_m\rangle \\ & & & \langle T,\theta_m\rangle & 0 \end{pmatrix}, \tag{7}$$

where $\langle\ ,\ \rangle$ *denotes the Euclidean inner product on* $\mathbb{R}^k$. *Moreover,*

$$\mathbb{R}^k \ltimes_{\rho_\theta} \mathbb{R}^{2m} \cong \mathbb{R}^k \ltimes_{\rho_{\theta'}} \mathbb{R}^{2m}$$

as flat Lie algebras if and only if ρ_θ *and* $\rho_{\theta'}$ *lie in the same* $\mathrm{O}(k)\times\mathrm{O}(2m)$-*orbit under the action* (6).

Remark. Note that the Lie algebra $\mathbb{R}^k \ltimes_{\rho_\theta} \mathbb{R}^{2m}$ is a Lie subalgebra of the Euclidean Lie algebra $\mathfrak{e}(2m)$:

$$\mathbb{R}^k \ltimes_{\rho_\theta} \mathbb{R}^{2m} \hookrightarrow \mathfrak{e}(2m), \qquad (T, W) \mapsto \begin{pmatrix} \rho_\theta(T) & W \\ 0 & 0 \end{pmatrix},$$

$T \in \mathbb{R}^k$, $W \in \mathbb{R}^{2m}$. However, the inner product on $\mathbb{R}^k \ltimes_{\rho_\theta} \mathbb{R}^{2m}$ does not coincide with the one induced from $\mathfrak{e}(2m)$.

The next corollary follows from the description given in Proposition 2.2.

Corollary 2.1. *Any even dimensional flat Lie algebra is Kähler flat.*

Proof. Let $\mathfrak{g}_\theta = \mathbb{R}^s \times (\mathbb{R}^k \ltimes_{\rho_\theta} \mathbb{R}^{2m})$ be as in Proposition 2.2 and let J be the orthogonal endomorphism of $\mathfrak{g}_\theta$, leaving $\mathbb{R}^s \times \mathbb{R}^k$ invariant and such that $J^2 = -\mathrm{id}$, $Jf_{2i+1} = f_{2i}$, $i = 0, \ldots, m-1$. The integrability of J, that is, the vanishing of N_J, follows from $\rho_\theta(T)J = J\rho_\theta(T)$, for any $T \in \mathbb{R}^k$. Moreover, $\nabla J = 0$ since $\nabla_T = \rho_\theta(T)$, for $T \in \mathbb{R}^k$. Therefore $(\mathfrak{g}, J, g)$ is Kähler flat. □

3. Hyper-Kähler Lie groups

In this section we shall apply Proposition 2.1 to give a characterization of the Lie algebras carrying a hyper-Kähler structure $(\{J_\alpha\}, g)$.

Proposition 3.1. *Let* $(\mathfrak{g}, \{J_\alpha\}, g)$, $\alpha = 1, 2, 3$, *be a hyper-Kähler Lie algebra. Then* $\mathfrak{g}$ *decomposes orthogonally as*

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{g}^1, \qquad \mathfrak{z}(\mathfrak{g}) \subset \mathfrak{t},$$

with both $\mathfrak{t}$ *and* $\mathfrak{g}^1$ *abelian and* J_α*-invariant,* $\alpha = 1, 2, 3$, *such that*

(i) $\mathrm{ad}_X J_\alpha = J_\alpha \mathrm{ad}_X$, *for any* $X \in \mathfrak{t}$, $\alpha = 1, 2, 3$;
(ii) $g(\mathrm{ad}_X Y, Z) + g(Y, \mathrm{ad}_X Z) = 0$, *for any* $X \in \mathfrak{t}$, $Y, Z \in \mathfrak{g}$.

Proof. Since a hyper-Kähler Lie algebra is flat [1], $\mathfrak{g}$ decomposes orthogonally as $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{h} \oplus \mathfrak{g}^1$ and the conditions in Proposition 2.1 are satisfied. Set

$$\mathfrak{t} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{h}.$$

We show next that if $(\mathfrak{g}, \{J_\alpha\}, g)$, $\alpha = 1, 2, 3$, is a hyper-Kähler Lie algebra then $\mathfrak{t}$ and $\mathfrak{g}^1$ are J_α-invariant, $\alpha = 1, 2, 3$, and condition (i) is satisfied. Observe that if $X \in \mathfrak{t}$ and $B \in \mathfrak{g}^1$, using that $\nabla J_\alpha = 0$ and (ii) of Proposition 2.1, one has

$$J_\alpha[X, B] = J_\alpha \nabla_X B = [X, J_\alpha B],$$

therefore, (i) follows. Since $\mathfrak{g}^1 = [\mathfrak{h}, \mathfrak{g}^1]$, the above equation also implies that $\mathfrak{g}^1$ is J_α-invariant and the decomposition $\mathfrak{t} \oplus \mathfrak{g}^1$ satisfies the desired properties. $\square$

We will say that two hyper-Kähler Lie algebras $(\mathfrak{g}, \{J_\alpha\}, g)$ and $(\mathfrak{g}', \{J'_\alpha\}, g')$ are equivalent if there exists an equivalence η of metric Lie algebras such that $\eta J_\alpha = J'_\alpha \eta, \alpha = 1, 2, 3$.

Consider the hypercomplex structure on

$$\mathbb{H}^q = \{(W_1, \ldots, W_q) : W_\alpha = u_\alpha + y_\alpha i + z_\alpha j + w_\alpha k : u_\alpha, y_\alpha, z_\alpha, w_\alpha \in \mathbb{R}\}$$

given by right multiplication by $-i, -j, -k$:

$$J_1 = R_{-i}, \qquad J_2 = R_{-j}, \qquad J_3 = R_{-k}.$$

We identify $\mathbb{H}^q \cong \mathbb{R}^{4q}$ with the Euclidean metric and we let $\mathrm{Sp}(q) = \mathrm{O}(4q) \cap \mathrm{GL}(q, \mathbb{H})$, where

$$\mathrm{GL}(q, \mathbb{H}) = \{T \in \mathrm{GL}(4q, \mathbb{R}) : TJ_\alpha = J_\alpha T, \alpha = 1, 2, 3\}.$$

Let $\mathfrak{t}^q$ be the following maximal abelian subalgebra of the Lie algebra $\mathfrak{sp}(q)$ of $\mathrm{Sp}(q)$:

$$\mathfrak{t}^q = \left\{ \begin{pmatrix} 0 & -\phi_1 & 0 & 0 & & & & & \\ \phi_1 & 0 & 0 & 0 & & & & & \\ 0 & 0 & 0 & -\phi_1 & & & & & \\ 0 & 0 & \phi_1 & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & -\phi_q & 0 & 0 \\ & & & & & \phi_q & 0 & 0 & 0 \\ & & & & & 0 & 0 & 0 & -\phi_q \\ & & & & & 0 & 0 & \phi_q & 0 \end{pmatrix} : \phi_i \in \mathbb{R} \right\}. \tag{8}$$

We obtain the analogue of Proposition 2.2 by arguing as before. Observe that, in this case, $\mathbb{R}^s \times (\mathbb{R}^k \ltimes_{\rho_\theta} \mathbb{H}^q) \cong \mathbb{R}^s \times (\mathbb{R}^k \ltimes_{\rho_{\theta'}} \mathbb{H}^q)$ as hyper-Kähler Lie algebras if and only if ρ_θ and $\rho_{\theta'}$ lie in the same $\mathrm{Sp}(p) \times \mathrm{Sp}(q)$-orbit, where $s + k = 4p$ and the action of $\mathrm{Sp}(p) \times \mathrm{Sp}(q)$ is the analogue of (6).

Proposition 3.2. *Let* $(\mathfrak{g}, \{J_\alpha\}, g)$ *be a hyper-Kähler Lie algebra with* $\dim \mathfrak{g}^1 = 4q$ *and* $\dim \mathfrak{z}(\mathfrak{g}) = s$. *Then there exists* $\theta = (\theta^\alpha_\beta) \in M(k, q; k)$, *with* $s + k = 4p$, *such that* $\theta_\beta \neq 0$ *for* $1 \leq \beta \leq q$ *and*

$$\mathfrak{g} \cong \mathbb{R}^s \times (\mathbb{R}^k \ltimes_{\rho_\theta} \mathbb{H}^q).$$

$\mathbb{R}^k \ltimes_{\rho_\theta} \mathbb{H}^q$ is the Lie algebra with orthonormal basis

$$\{e_r, 1 \le r \le k, f_a, f_a i, f_a j, f_a k, 1 \le a \le q\}$$

such that $T \in \mathbb{R}^k$ acts on $\mathbb{H}^q$ by

$$\rho_\theta(T) = \begin{pmatrix} \rho_\theta^1(T) & & \\ & \ddots & \\ & & \rho_\theta^q(T) \end{pmatrix}, \tag{9}$$

where

$$\rho_\theta^\beta(T) = \begin{pmatrix} 0 & -\langle T, \theta_\beta \rangle & 0 & 0 \\ \langle T, \theta_\beta \rangle & 0 & 0 & 0 \\ 0 & 0 & 0 & -\langle T, \theta_\beta \rangle \\ 0 & 0 & \langle T, \theta_\beta \rangle & 0 \end{pmatrix}$$

and $\langle \, , \rangle$ denotes the Euclidean inner product on $\mathbb{R}^k$. The Lie algebra $\mathbb{R}^s \times (\mathbb{R}^k \ltimes_{\rho_\theta} \mathbb{H}^q)$ is hyper-Kähler with its natural hypercomplex structure, obtained by extending R_{-i}, R_{-j}, R_{-k} on $\mathbb{H}^q$ by any triple of complex endomorphisms on $\mathbb{R}^s \times \mathbb{R}^k$ satisfying the quaternion relations, and the canonical inner product. Moreover,

$$\mathbb{R}^s \times (\mathbb{R}^k \ltimes_{\rho_\theta} \mathbb{H}^q) \cong \mathbb{R}^s \times (\mathbb{R}^k \ltimes_{\rho_{\theta'}} \mathbb{H}^q)$$

as hyper-Kähler Lie algebras if and only if ρ_θ and $\rho_{\theta'}$ lie in the same $\mathrm{Sp}(p) \times \mathrm{Sp}(q)$*-orbit.*

3.1. Examples

As a consequence of Proposition 3.2 we have that there is a one parameter family of eight-dimensional hyper-Kähler Lie algebras $\mathfrak{g}_\theta$:

$$\mathfrak{g}_\theta \cong \mathbb{R}^3 \times (\mathbb{R} \ltimes_\theta \mathbb{H}), \tag{10}$$

where $\mathbb{R} \ltimes_\theta \mathbb{H}$ has an orthonormal basis $\{e_1, f_1, f_1 i, f_1 j, f_1 k\}$ and e_1 acts on $\mathbb{H}$ as follows:

$$\rho_\theta(e_1) = \begin{pmatrix} 0 & -\theta & & \\ \theta & 0 & & \\ & & 0 & -\theta \\ & & \theta & 0 \end{pmatrix}.$$

Note that these are pairwise non equivalent flat metric Lie algebras for different values of θ, but they are isomorphic as Lie algebras for $\theta \neq 0$.

In dimension 12 there are infinitely many non isomorphic Lie algebra structures admitting hyper-Kähler metrics. In fact, for a fixed real number $s \neq 0$ we define $\mathfrak{g}_s = \mathbb{R}^3 \times (\mathbb{R} \ltimes_s \mathbb{H}^2)$, where $\mathbb{R} \ltimes_s \mathbb{H}^2$ has an orthonormal basis as in the statement of Proposition 3.2 with e_1 acting on $\mathbb{H}^2$ as follows:

$$\rho_s(e_1) = \begin{pmatrix} 0 & -1 & & & & & & \\ 1 & 0 & & & & & & \\ & & 0 & -1 & & & & \\ & & 1 & 0 & & & & \\ & & & & 0 & -s & & \\ & & & & s & 0 & & \\ & & & & & & 0 & -s \\ & & & & & & s & 0 \end{pmatrix}.$$

It turns out that $\mathfrak{g}_s$ and $\mathfrak{g}_r$ are non isomorphic for $s \neq r$.

We describe now the Lie bracket on $\mathfrak{g}_\theta = \mathbb{R}^k \ltimes_{\rho_\theta} \mathbb{H}^q$:

$$\begin{aligned}[] [(X, W), (X', W')] &= (0, i(\langle X, \theta_1\rangle W'_1 - \langle X', \theta_1\rangle W_1), \dots, i(\langle X, \theta_q\rangle W'_q - \langle X', \theta_q\rangle W_q)) \\ &= (0, \rho_\theta(X)W' - \rho_\theta(X')W), \end{aligned} \tag{11}$$

$X, X' \in \mathbb{R}^k$.

The product on the simply connected Lie group $G_\theta = \mathbb{R}^k \ltimes_\theta \mathbb{H}^q$ with Lie algebra $\mathfrak{g}_\theta$ is given as follows:

$$(X, W) \cdot (X', W') = (X + X',\ W + \theta(X)W'), \tag{12}$$

where $X, X' \in \mathbb{R}^k,\ W, W' \in \mathbb{H}^q$, $W' = (W'_1, \dots, W'_q)$ and

$$\theta(X)W' = (\mathrm{e}^{\mathrm{i}\langle X,\theta_1\rangle} W'_1, \dots, \mathrm{e}^{\mathrm{i}\langle X,\theta_q\rangle} W'_q). \tag{13}$$

Using that $(X, W)^{-1} = -(X, \theta(-X)W)$, conjugation by (X, W) is given as follows:

$$I_{(X,W)}(X', W') = (X, W) \cdot (X', W') \cdot (X, W)^{-1} = (X', W + \theta(X)W' - \theta(X')W)$$

and therefore

$$\mathrm{Ad}(X, W)(X', W') = (X', \theta(X)W') + [(0, W), (X', 0)] = (X', \theta(X)W' - \rho_\theta(X')W), \tag{14}$$

for $X, X' \in \mathbb{R}^k$, $W, W' \in \mathbb{H}^q$.

In coordinates $(x_1, \dots, x_k, W_1, \dots, W_q)$, where $W_j = (u_j, y_j, z_j, w_j)$, the left invariant flat metric g on $\mathbb{R}^k \ltimes_{\rho_\theta} \mathbb{H}^q$ is the Euclidean metric

$$g = \sum_{j=1}^{k} \mathrm{d}x_j^2 + \sum_{j=1}^{q} (\mathrm{d}u_j^2 + \mathrm{d}y_j^2 + \mathrm{d}z_j^2 + \mathrm{d}w_j^2).$$

For the constructions of the next section, we will need to express the Euclidean metric on $\mathbb{H}^q$ in suitable coordinates. Any quaternion may be written as

$$W_\beta = \mathrm{e}^{\mathrm{i}\psi_\beta/2} a_\beta, \quad \beta = 1, \ldots, q,$$

with $\psi_\beta \in (0, 4\pi]$ and a_β a pure imaginary quaternion, so that $\bar{a}_\beta = -a_\beta$. Let

$$\mathbf{r}_\beta = \bar{W}_\beta i W_\beta = \bar{a}_\beta i a_\beta = -a_\beta i a_\beta.$$

The flat metric on $\mathbb{H}^q$ in coordinates $(\psi_\beta, \mathbf{r}_\beta)$, $\beta = 1, \ldots, q$, is given by

$$\frac{1}{4}\sum_{\beta=1}^{q}\left(\frac{1}{r_\beta}\,\mathrm{d}\mathbf{r}_\beta^2 + r_\beta(\mathrm{d}\psi_\beta + \mathbf{\Omega}_\beta \cdot \mathrm{d}\mathbf{r}_\beta)^2\right), \tag{15}$$

where

$$r_\beta = |\mathbf{r}_\beta|, \qquad \mathrm{curl}(\mathbf{\Omega}_\beta) = \mathrm{grad}\left(\frac{1}{r_\beta}\right)$$

(the curl and grad operations are taken with respect to the Euclidean metric on $\mathbb{R}^3$ with cartesian coordinates $\mathbf{r}_\beta$).

4. Main properties of the hyper-Kähler quotient metrics

According to Proposition 3.2 any simply connected Lie group with a left invariant hyper-Kähler structure is of the form $G_\theta = \mathbb{R}^s \times (\mathbb{R}^k \ltimes_\theta \mathbb{H}^q)$ $(k \le q, s + k = 4p)$ with the hyper-Kähler metric $g = g_1 \times g_2$, where g_1 is the Euclidean metric on $\mathbb{R}^s \times \mathbb{R}^k$ and g_2 is the Euclidean metric on $\mathbb{H}^q$. Let $\mathfrak{g}_\theta$ be the Lie algebra of G_θ. The associated Kähler forms:

$$\omega_\alpha((X_1, W_1), (X_2, W_2)) = g(J_\alpha(X_1, W_1), (X_2, W_2)), \qquad (X_1, W_1), (X_2, W_2) \in \mathfrak{g}_\theta,$$

$\alpha = 1, 2, 3$, when left translated to G_θ become:

$$\omega_\alpha = \omega_\alpha^1 + \omega_\alpha^2,$$

where ω_α^j, $j = 1, 2$, $\alpha = 1, 2, 3$, are the standard symplectic forms on a vector space. Therefore, $(G_\theta, g, \omega_\alpha)$ is equivalent, as a hyper-Kähler manifold, to the product

$$(\mathbb{R}^s \times \mathbb{R}^k, g_1, \{\omega_\alpha^1\}) \times (\mathbb{H}^q, g_2, \{\omega_\alpha^2\}). \tag{16}$$

We will apply the hyper-Kähler quotient construction in [8] to the case when L is the connected closed abelian Lie subgroup $\mathbb{R}^l \subset \mathbb{R}^k$ with Lie algebra $\mathfrak{l} = \mathrm{span}_{\mathbb{R}}\{e_1, \ldots, e_l\}$ such that $\mathfrak{l}$ is isotropic with respect to ω_α for each α. The action of L on G_θ will be given by left translations, therefore it preserves the hyper-Kähler structure. We recall next the quotient construction in our particular case.

Let $\mathcal{X}_V$ be the vector field generated by the action of L, that is, the right invariant vector field such that $\mathcal{X}_{V_e} = V$, where $V \in \mathfrak{l}$. Observe that

$$0 = L_{\mathcal{X}_V}\omega_\alpha = \mathrm{d}(i(\mathcal{X}_V)\omega_\alpha) + i(\mathcal{X}_V)\,\mathrm{d}\omega_\alpha,$$

where $i(\mathcal{X}_V)\omega_\alpha$ denotes the 1-form obtained by taking the interior product with $\mathcal{X}_V$. Since the action is symplectic with respect to ω_α, $\alpha = 1, 2, 3$, we have that $i(\mathcal{X}_V)\omega_\alpha$, $\alpha = 1, 2, 3$, is closed. G_θ is simply connected, thus $H^1_{\mathrm{d}R}(G_\theta, \mathbb{R}) = \{0\}$ and $i(\mathcal{X}_V)\omega_\alpha$ is exact, that is,

$$i(\mathcal{X}_V)\omega_\alpha = \mathrm{d}(\mu_\alpha^\theta)^V,$$

where $(\mu_\alpha^\theta)^V$ is a Hamiltonian function associated to V. Putting all these functions together, we obtain a map into the dual space of the Lie algebra of L

$$\mu_\alpha^\theta : G_\theta \to \mathfrak{l}^*$$

defined by

$$\mu_\alpha^\theta(X, W)(V) = (\mu_\alpha^\theta)^V(X, W).$$

There is a choice of constants in the definition of μ_α^θ, since each function $(\mu_\alpha^\theta)^V$ is determined up to an additive constant. When the ambiguities in the choices of $(\mu_\alpha^\theta)^V$ can be adjusted to make μ_α^θ L-equivariant, where L acts on $\mathfrak{l}^*$ by the coadjoint action, one has the hyper-Kähler moment map

$$\mu^\theta : G_\theta \to \mathfrak{l}^* \otimes \mathrm{Im}\,\mathbb{H},$$

defined by $\mu^\theta = \mu_1^\theta i + \mu_2^\theta j + \mu_3^\theta k$. Our choice of L implies that μ_α^θ is L-equivariant for each α. Indeed, the action A of L on G_θ given by left translations:

$$A : L \times G_\theta \to G_\theta, \qquad ((V, 0), (X, W)) \to (V, 0) \cdot (X, W) = (V + X, \theta(V)W) \tag{17}$$

(recall (12)) can be viewed as a diagonal action of L:

$$A(V)(X, W) = (A_1(V)X, A_2(V)W),$$

where A_1 acts by left translations on $\mathbb{R}^s \times \mathbb{R}^k$ and A_2 is a linear symplectic action on $\mathbb{H}^q$. The moment map μ_α^θ corresponding to A can be obtained by adding up the moment maps of A_1 and A_2 since (16) holds. By a direct calculation one has:

$$\mu_\alpha^\theta(X, W)(V) = \omega_\alpha(V, X) + \frac{1}{2}\omega_\alpha(\rho_\theta(V)W, W).$$

The L-equivariance of the first term follows since L is isotropic and the second term is L-equivariant since it is the moment map of a linear action on a symplectic vector space (see [6]).

Let $\xi \in \mathfrak{z} \otimes \mathbb{R}^3$ be a regular value for μ^θ, where $\mathfrak{z}$ is the subspace of $\mathfrak{l}^*$ of invariant elements under the coadjoint action, and consider the quotient space, $L\backslash(\mu^\theta)^{-1}(\xi)$.

Our hypotheses imply:

(1) $\mathfrak{z}$ coincides with $\mathfrak{l}^*$ since L is abelian;
(2) the action of L on G_θ is free, hence it is free on $(\mu^\theta)^{-1}(\xi)$, for any $\xi = \xi_1 i + \xi_2 j + \xi_3 k$ in the image $\operatorname{Im}\mu^\theta$ of μ^θ. In particular, any $\xi \in \operatorname{Im}\mu^\theta$ is a regular value of the hyper-Kähler moment map;
(3) since L is closed in G_θ and acts by left translations, the set of right cosets $L\backslash G_\theta$ is a complete Riemannian manifold, not necessarily homogeneous.

In the next theorem we show that $L\backslash(\mu^\theta)^{-1}(\xi)$ is a Hausdorff manifold for any $\xi \in \operatorname{Im}\mu^\theta$, therefore, according to [8], the hyper-Kähler metric on G_θ induces a hyper-Kähler metric on $L\backslash(\mu^\theta)^{-1}(\xi)$. We also state the main properties of the resulting metrics.

Theorem 4.1. *Let $(G_\theta, \{J_\alpha\}, g)$ be a simply connected hyper-Kähler Lie group, so that $G_\theta = \mathbb{R}^s \times (\mathbb{R}^k \ltimes_\theta \mathbb{H}^q)$, $k \le q$, $s + k = 4p$. Fix a connected closed abelian isotropic subgroup $L \subset \mathbb{R}^k$ acting on G_θ by the action A as in (17) and denote by $\pi : G_\theta \to L\backslash G_\theta$ the associated Riemannian submersion. Then*

(1) *The action A of L on G_θ is free and preserves both, the metric g and the symplectic forms ω_α, $\alpha = 1, 2, 3$. The L-equivariant moment map is $\mu^\theta = \mu_1^\theta i + \mu_2^\theta j + \mu_3^\theta k$, with μ_α^θ given by*

$$\mu_\alpha^\theta(X, W)(V) = \omega_\alpha(V, X) + \frac{1}{2}\omega_\alpha(\rho_\theta(V)W, W),$$

for any $X \in \mathbb{R}^s \times \mathbb{R}^k$, $W \in \mathbb{H}^q$, $V \in \mathfrak{l}$.
(2) *$L\backslash G_\theta$ is a complete Riemannian manifold of non negative sectional curvature. Moreover, the fibers of π are totally geodesic;*
(3) *For any $\xi \in \operatorname{Im}\mu^\theta$, $L\backslash(\mu^\theta)^{-1}(\xi)$ is a closed embedded submanifold of $L\backslash G_\theta$;*
(4) *The metric on $L\backslash(\mu^\theta)^{-1}(\xi)$ which makes π a Riemannian submersion coincides with the restriction of the given one in $L\backslash G_\theta$. In particular, the hyper-Kähler metric on the quotient $L\backslash(\mu^\theta)^{-1}(\xi)$ is complete.*

Proof. The proof of part (1) was done in the paragraph containing Eq. (17).

The left invariant metric g on G_θ induces in a natural way a metric $\tilde{g}$ on $L\backslash G_\theta$ such that the natural projection

$$\pi : (G_\theta, g) \to (L\backslash G_\theta, \tilde{g})$$

is a Riemannian submersion. The completeness of g implies that $\tilde{g}$ is also complete (see [7]). By O'Neill's formula ([14, 3, Corollary 1]), the sectional curvature of $L\backslash G_\theta$ is non negative. Note that $\mathfrak{l} \subset [\mathfrak{g}_\theta, \mathfrak{g}_\theta]^\perp$, hence the fibers of π are totally geodesic since $\nabla_T V = 0$ for $T, V \in \mathfrak{l}$ (see Proposition 2.1). This proves part (2).

If $\xi \in \operatorname{Im} \mu^\theta$, then ξ is a regular value of μ^θ and since μ^θ is L-equivariant, it induces a map

$$\tilde{\mu}^\theta : L\backslash G_\theta \to \mathfrak{l}^* \otimes \operatorname{Im} \mathbb{H}.$$

We have $L\backslash(\mu^\theta)^{-1}(\xi) = (\tilde{\mu}^\theta)^{-1}(\xi)$, and ξ is a regular value of $\tilde{\mu}^\theta$, therefore $L\backslash(\mu^\theta)^{-1}(\xi)$ is a closed d-dimensional embedded submanifold of $L\backslash G_\theta$ ($d = \dim\, G_\theta - 4 \dim\, L$), and part (3) follows.

The first claim of part (4) follows from the fact that the metric on $(\mu^\theta)^{-1}(\xi)$ is the one induced from (G_θ, g) and by observing that π is a Riemannian submersion. This, together with parts (2) and (3), implies that the induced hyper-Kähler metric on the quotient $L\backslash(\mu^\theta)^{-1}(\xi)$ is complete. □

5. Examples

In the next examples we show that it is possible to describe several known families of hyper-Kähler metrics [4] in a unified way, by applying the quotient construction to hyper-Kähler Lie groups G under the action of a suitable closed abelian subgroup L by left translations.

5.1. Taub-Nut metric

Let $\mathfrak{g}_\theta$ be the one parameter family of hyper-Kähler Lie algebras in dimension 8 (see (10)) and G_θ the corresponding simply connected Lie groups. Let $L \simeq \mathbb{R}$ be the subgroup of $G_\theta = \mathbb{H} \ltimes_\theta \mathbb{H}$ given by $L = \{(t, 0),\ t \in \mathbb{R}\}$, acting on G_θ by left translations, that is:

$$L \times G_\theta \to G_\theta, \qquad (t, (q, w)) \to (t + q, \mathrm{e}^{\mathrm{i}\theta t} w).$$

Observe that $\operatorname{Im} \mathbb{H}$ acts trivially on the second factor. The corresponding hyper-Kähler moment map is

$$\mu^\theta = -\operatorname{Im}(q) - \frac{\theta}{2}(\operatorname{Re}(iwi\bar{w})i + \operatorname{Re}(iwj\bar{w})j + \operatorname{Re}(iwk\bar{w})k) = -\operatorname{Im}(q) + \frac{\theta}{2}\bar{w}iw.$$

It can be checked that μ^θ is L-equivariant. The complete hyper-Kähler metric on $L\backslash(\mu^\theta)^{-1}(0)$ is the Taub-Nut metric with parameter θ^{-1} [4].

5.2. Generalized Taubian-Calabi metric

Let $\theta = (\theta_1, \dots, \theta_m) \in \mathbb{R}^m,\ G_\theta = \mathbb{H} \ltimes_\theta \mathbb{H}^m$ and $L = \{(t, 0) : t \in \mathbb{R}\}$ acting on G_θ by left translations:

$$L \times G_\theta \to G_\theta, \qquad (t, (q, w_1, \dots, w_m)) \to (t + q, \mathrm{e}^{\mathrm{i}\theta_1 t} w_1, \dots, \mathrm{e}^{\mathrm{i}\theta_m t} w_m).$$

For $m = 1$ this is the Lie group considered in the first example. The corresponding hyper-Kähler moment map is

$$\mu^\theta = -\mathrm{Im}(q) - \frac{1}{2}\sum_{\beta=1}^{m} \theta_\beta(\mathrm{Re}(iw_\beta i\overline{w_\beta})i + \mathrm{Re}(iw_\beta j\overline{w_\beta})j + \mathrm{Re}(iw_\beta k\overline{w_\beta})k)$$
$$= -\,\mathrm{Im}(q) + \frac{1}{2}\sum_{\beta=1}^{m} \theta_\beta \overline{w_\beta} i w_\beta.$$

When $\theta_\beta = 1$, for each β, the complete hyper-Kähler metric on $L\backslash(\mu^\theta)^{-1}(0)$ coincides with the Taubian-Calabi metric [16,4].

5.3. Lee–Weinberg–Yi metric

Let $\theta \in \mathrm{GL}(m, \mathbb{R}),\ G_\theta = \mathbb{H}^m \ltimes_\theta \mathbb{H}^m$ and $L \simeq \mathbb{R}^m$ defined by

$$L = \{((t_1, \dots, t_m), 0) : t_i \in \mathbb{R}\}$$

acting on G_θ by left translations:

$$L \times G_\theta \to G_\theta, \qquad ((t_1, \dots, t_m), (q_1, \dots, q_m, w_1, \dots, w_m))$$
$$\to (t_1 + q_1, \dots, t_m + q_m, \mathrm{e}^{\mathrm{i}\langle\theta_1, T\rangle} w_1, \dots, \mathrm{e}^{\mathrm{i}\langle\theta_m, T\rangle} w_m),$$

where $T = (t_1, \dots, t_m)$, θ_β are the rows of θ and $\langle\, , \rangle$ is the Euclidean inner product in $\mathbb{R}^m$. The corresponding hyper-Kähler moment map is

$$\mu^\theta = \left(-\mathrm{Im}(q_1) + \frac{1}{2}\sum_{\beta=1}^{m} \theta_\beta^1 \overline{w_\beta} i w_\beta, \dots, -\mathrm{Im}(q_m) + \frac{1}{2}\sum_{\beta=1}^{m} \theta_\beta^m \overline{w_\beta} i w_\beta\right).$$

The complete hyper-Kähler metric on $L\backslash(\mu^\theta)^{-1}(0)$ is the Lee–Weinberg–Yi metric with $(\lambda_b^a) = \theta^{-1}$ [11,13,4].

6. Topology of the quotient and local description of the metrics

Let G_θ be the simply connected hyper-Kähler Lie group $\mathbb{R}^s \times (\mathbb{R}^k \ltimes_\theta \mathbb{H}^q)$ $(s + k = 4p,$ $k \le q)$, $\theta \in M(k, q; k)$ and $\theta(\mathbb{H}^q) = \mathbb{H}^q$. Let $L \subset \mathbb{R}^k$ be a closed abelian subgroup with Lie algebra $\mathfrak{l} = \mathrm{span}_{\mathbb{R}}\{e_1, \dots, e_l\}$ such that $\mathfrak{l}$ is isotropic with respect to ω_α, for each α.

Let T^q be the maximal torus of Sp(q) with Lie algebra $\mathfrak{t}^q$ (see (8)) whose elements are of the form:

$$B = \begin{pmatrix} B(\phi_1) & & \\ & \ddots & \\ & & B(\phi_q) \end{pmatrix}, \tag{18}$$

where $\phi_\beta \in \mathbb{R}$ and $B(\phi_\beta)$ is the following 4×4 real matrix:

$$\begin{pmatrix} \cos(\phi_\beta) & -\sin(\phi_\beta) & 0 & 0 \\ \sin(\phi_\beta) & \cos(\phi_\beta) & 0 & 0 \\ 0 & 0 & \cos(\phi_\beta) & -\sin(\phi_\beta) \\ 0 & 0 & \sin(\phi_\beta) & \cos(\phi_\beta) \end{pmatrix}.$$

We have an action φ of T^q on G_θ:

$$\varphi : T^q \times G_\theta \to G_\theta, \qquad (B, (X, W)) \to \varphi(B, (X, W)) = (X, BW), \tag{19}$$

where BW stands for the product of the $4q \times 4q$ matrix B given in (18) by the column vector $W \in \mathbb{H}^q \cong \mathbb{R}^{4q}$. Note that the action φ commutes with A (see (17)) and both, A and φ, preserve the metric and are tri-holomorphic. Therefore, T^q also acts on the hyper-Kähler quotient $L\backslash(\mu^\theta)^{-1}(\xi)$ by tri-holomorphic isometries.

In the next theorem we give the explicit description of the moment map together with properties of the T^q-action.

Theorem 6.1. *Let $G_\theta = \mathbb{R}^s \times (\mathbb{R}^k \ltimes_\theta \mathbb{H}^q)$ be a hyper-Kähler Lie group, $s + k = 4p$, $\theta \in M(k, q; k)$, L the connected closed abelian isotropic subgroup $L \subset \mathbb{R}^k$ defined above and A, φ as in (17) and (19). Then*:

(1) *The expression of the moment map is*

$$\mu^\theta(X, W) = \left(-\mathrm{Im}\, X_1 + \frac{1}{2}\sum_{\beta=1}^{q} \theta_\beta^1 \overline{W_\beta} i W_\beta, \ldots, -\mathrm{Im}\, X_l + \frac{1}{2}\sum_{\beta=1}^{q} \theta_\beta^l \overline{W_\beta}\, i W_\beta \right), \tag{20}$$

for $(X, W) \in G_\theta$.

(2) *We have the following diffeomorphisms*:

$$L\backslash(\mu^\theta)^{-1}(\xi) \cong \mathbb{R}^{4p+4q-4l}, \ \textit{for any}\ \xi \in \mathrm{Im}\,\mu^\theta, \qquad L\backslash G_\theta \cong \mathbb{R}^{4p+4q-l}.$$

(3) *The torus T^q acts on $L\backslash(\mu^\theta)^{-1}(\xi)$ by tri-holomorphic isometries. If $l = p = q$, the action of T^q on the $4q$-dimensional quotient has a unique fixed point.*

Proof. We start by proving part (2). In order to do it we will find global coordinates on $L\backslash(\mu^\theta)^{-1}(\xi)$ and $L\backslash G_\theta$. For $(X, W) \in G_\theta$, $(T, 0) \in L$, set

$$X = \sum_{\alpha=1}^{p} e_\alpha(x_\alpha + b_\alpha i + s_\alpha j + p_\alpha k), \quad T = \sum_{\alpha=1}^{l} t_\alpha e_\alpha, \tag{21}$$

$$W = \sum_{\alpha=1}^{q} f_\alpha(u_\alpha + y_\alpha i + z_\alpha j + w_\alpha k). \tag{22}$$

It follows that $(x_\alpha, b_\gamma, s_\gamma, p_\gamma, u_\beta, y_\beta, z_\beta, w_\beta)$, with $\alpha = l+1, \ldots, p$, $\gamma = l, \ldots, p$, $\beta = 1, \ldots, q$, are global coordinates on $L\backslash G_\theta$ and therefore $L\backslash G_\theta$ is diffeomorphic to $\mathbb{R}^{4p+4q-l}$. Using the fact that the hypercomplex structure corresponds to

$$J_1 = R_{-i}, \qquad J_2 = R_{-j}, \qquad J_3 = R_{-k}$$

and that the metric g is such that the real basis

$$\{e_\alpha, e_\alpha i, e_\alpha j, e_\alpha k, f_\beta, f_\beta i, f_\beta j, f_\beta k, 1 \le \alpha \le p,\ 1 \le \beta \le q\}$$

is orthonormal, we get, using Theorem 4.1, the following expressions for the moment maps μ^θ_γ, $\gamma = 1, 2, 3$, in terms of the real coordinates on $\mathbb{H}^p$ and $\mathbb{H}^q$:

$$\begin{aligned}
\mu^\theta_1(X, W)(T) &= -\sum_{\alpha=1}^{l} b_\alpha t_\alpha + \frac{1}{2}\sum_{\alpha=1}^{l} t_\alpha \left(\sum_{\beta=1}^{q} \theta^\alpha_\beta(u_\beta^2 + y_\beta^2 - z_\beta^2 - w_\beta^2)\right)\\
&= g\left(T, \sum_{\alpha=1}^{l}\left(-b_\alpha + \frac{1}{2}\sum_{\beta=1}^{q}\theta^\alpha_\beta(u_\beta^2 + y_\beta^2 - z_\beta^2 - w_\beta^2)\right) e_\alpha\right),\\
\mu^\theta_2(X, W)(T) &= -\sum_{\alpha=1}^{l} s_\alpha t_\alpha + \sum_{\alpha=1}^{l} t_\alpha \left(\sum_{\beta=1}^{q} \theta^\alpha_\beta(-u_\beta w_\beta + z_\beta y_\beta)\right)\\
&= g\left(T, \sum_{\alpha=1}^{l}\left(-s_\alpha + \sum_{\beta=1}^{q}\theta^\alpha_\beta(-u_\beta w_\beta + z_\beta y_\beta)\right) e_\alpha\right),\\
\mu^\theta_3(X, W)(T) &= -\sum_{\alpha=1}^{l} p_\alpha t_\alpha + \sum_{\alpha=1}^{l} t_\alpha \left(\sum_{\beta=1}^{q} \theta^\alpha_\beta(u_\beta z_\beta + w_\beta y_\beta)\right)\\
&= g\left(T, \sum_{\alpha=1}^{l}\left(-p_\alpha + \sum_{\beta=1}^{q}\theta^\alpha_\beta(u_\beta z_\beta + w_\beta y_\beta)\right) e_\alpha\right),
\end{aligned}$$

or, equivalently, (20) holds and part (1) follows. On $(\mu^\theta)^{-1}(\xi)$ one has the following relations:

$$b_\alpha + (\xi_1)_\alpha = \frac{1}{2}\sum_{\beta=1}^{q} \theta^\alpha_\beta (u_\beta^2 + y_\beta^2 - z_\beta^2 - w_\beta^2),$$

$$s_\alpha + (\xi_2)_\alpha = -\sum_{\beta=1}^{q} \theta^\alpha_\beta (u_\beta w_\beta - z_\beta y_\beta), \qquad p_\alpha + (\xi_3)_\alpha = \sum_{\beta=1}^{q} \theta^\alpha_\beta (u_\beta z_\beta + w_\beta y_\beta),$$

for any $\alpha = 1 \ldots, l$, where $\xi_j = \sum_{\alpha=1}^{l} (\xi_j)_\alpha e_\alpha$, $j = 1, 2, 3$, and we think of $\xi = \xi_1 i + \xi_2 j + \xi_3 k$ as an element of $\mathfrak{l} \otimes \operatorname{Im}\mathbb{H}$ by means of the identification between $\mathfrak{l}$ and $\mathfrak{l}^*$ given by the restriction of g_1 to $\mathfrak{l}$. Thus, one has that $(x_\alpha, b_\gamma, s_\gamma, p_\gamma, u_\beta, y_\beta, z_\beta, w_\beta)$, with $\alpha = 1, \ldots, p$, $\gamma = l+1, \ldots, p$, $\beta = 1, \ldots, q$, are global coordinates on $(\mu^\theta)^{-1}(\xi)$.

Since the action of $\mathbb{R}^l$ leaves $x_\gamma, b_\gamma, s_\gamma, p_\gamma$, $\gamma \geq l+1$, invariant, and rotates the coordinates $u_\beta, y_\beta, z_\beta, w_\beta$, $\beta = 1, \ldots, q$, one has that $(x_\gamma, b_\gamma, s_\gamma, p_\gamma, u_\beta, y_\beta, z_\beta, w_\beta)$, with $\gamma = l+1, \ldots, p$, $\beta = 1, \ldots, q$, are global coordinates on $L\backslash(\mu^\theta)^{-1}(\xi)$. It follows that the quotient space is diffeomorphic to $\mathbb{R}^{4p+4q-4l}$.

For part (3), it follows from (19) that:

$$\varphi(B)(X, W) = (X, BW),$$

$B \in T^q$, $(X, W) \in G_\theta$. Since the moment map μ^θ satisfies

$$\mu^\theta(\varphi(B)(X, W)) = \mu^\theta(X, W),$$

then φ preserves $(\mu^\theta)^{-1}(\xi)$. In particular, the hyper-Kähler quotient admits a tri-holomorphic action of the torus T^q.

Assume next that $l = p = q$, hence $\mathfrak{l} \oplus J_1\mathfrak{l} \oplus J_2\mathfrak{l} \oplus J_3\mathfrak{l} = \mathbb{R}^{4p}$. Set $\xi = \xi_1 i + \xi_2 j + \xi_3 k$, where

$$\xi_j = \sum_{\alpha=1}^{l} (\xi_j)_\alpha e_\alpha \in \mathfrak{l}, \qquad j = 1, 2, 3.$$

We will consider $\xi \in \mathbb{R}^{4p}$ by means of the inclusion $\xi \hookrightarrow J_1\xi_1 + J_2\xi_2 + J_3\xi_3$. Let π be the natural projection from $(\mu^\theta)^{-1}(\xi)$ onto $L\backslash(\mu^\theta)^{-1}(\xi)$. If $(X, W) \in (\mu^\theta)^{-1}(\xi)$, then $\pi(X, W)$ is a fixed point for the action of T^q if and only if

$$(V, 0) \cdot (X, W) \cdot (V, 0)^{-1} \cdot (X, W)^{-1} \in L$$

for every $V \in \mathbb{R}^q$. We will show that $\pi(X, W) = \pi(\xi, 0)$, that is, $\pi(\xi, 0)$ is the unique fixed point. Using (19) and (12) we calculate

$$\begin{aligned}(V, 0) \cdot (X, W) \cdot (V, 0)^{-1} \cdot (X, W)^{-1} &= (V, 0) \cdot (-V, W - (-V) \cdot W) \\ &= (0, V \cdot W - W)\end{aligned}$$

which lies in L if and only if $V \cdot W = W$ for every $V \in \mathbb{R}^q$, hence $W = 0$. Since $(X, 0) \in (\mu^\theta)^{-1}(\xi)$ it follows that $\omega_\alpha(V, X) = g(\xi_\alpha, V)$ for every $V \in \mathfrak{l}$, $\alpha = 1, 2, 3$. Since $\mathfrak{l} \oplus J_1\mathfrak{l} \oplus J_2\mathfrak{l} \oplus J_3\mathfrak{l} = \mathbb{R}^{4p}$, the $\mathfrak{l}$ component of $J_\alpha X$ is ξ_α, or, equivalently, $X + \xi \in \mathfrak{l}$. Therefore $\pi(X, 0) = \pi(\xi, 0)$, as asserted. □

Using the fact that the quotient admits a tri-holomorphic T^q-action, we can obtain the local expression of the hyper-Kähler metric in terms of the structure constants of the Lie group G_θ.

Observe that, if $l = p$, the quotient has dimension $4q$, thus by [10,15] the induced hyper-Kähler metric can be locally written as follows:

$$\frac{1}{4}H_{\beta\gamma}\mathrm{d}\mathbf{r}_\beta \cdot \mathrm{d}\mathbf{r}_\gamma + \frac{1}{4}H^{\beta\gamma}(\mathrm{d}\tau_\beta + \mathbf{\Omega}_{\beta\delta} \cdot \mathrm{d}\mathbf{r}_\delta)(\mathrm{d}\tau_\gamma + \mathbf{\Omega}_{\gamma\epsilon} \cdot \mathrm{d}\mathbf{r}_\epsilon), \tag{23}$$

where $\beta, \gamma = 1, \ldots, q$, $(H^{\beta\gamma})$ is the inverse of the matrix $(H_{\beta\gamma})$. The Killing vector fields $\frac{\partial}{\partial\tau_\beta}$ generate the T^q-action, $\psi_\beta, \mathbf{r}_\beta$ are defined as in (15) and we use the Einstein summation convention. If $l < p$, the quotient splits as Riemannian product of the flat Euclidean space $\mathbb{R}^{4p-4l}$ by a $4q$-dimensional hyper-Kähler manifold with a tri-holomorphic T^q-action.

Theorem 6.2. *The local expression of the hyper-Kähler metric on the quotient $L\backslash(\mu^\theta)^{-1}(\xi)$ has the form $h = h_0 + h_1$, where h_0 is the Euclidean metric on $\mathbb{R}^{4p-4l}$ and h_1 is given by (23), with*

$$H_{\beta\gamma} = (\tilde{\theta}\tilde{\theta}^t)_{\beta\gamma} + \frac{1}{r_\beta}\,\delta_{\beta\gamma}. \tag{24}$$

$\tilde{\theta}$ is the $q \times l$ matrix obtained from θ by deleting the last $p - l$ columns, $\tilde{\theta}^t$ is its transpose and $r_\beta = |\mathbf{r}_\beta|$.

Proof. The action A of L on G_θ (recall (17)) in the coordinates $(x_\alpha, b_\alpha, s_\alpha, p_\alpha, \psi_\beta, \mathbf{r}_\beta)$ is given by

$$L \times G_\theta \to G_\theta, \quad (T, (X_\alpha, \psi_\beta, \mathbf{r}_\beta)) \to (X_\alpha + t_\alpha, \psi_\beta + 2\langle\theta_\beta, T\rangle, \mathbf{r}_\beta),$$

with $\alpha = 1, \ldots, p$, $t_\alpha = 0$ for $\alpha > l$, $\beta = 1, \ldots, q$ and $\psi_\beta, \mathbf{r}_\beta$ are defined as in (15).

The previous action leaves

$$\tau_\beta = \psi_\beta - 2\sum_{\alpha=1}^{p}\theta_\beta^\alpha x_\alpha, \quad \beta = 1, \ldots, q,$$

invariant and $\frac{\partial}{\partial\tau_\beta}$ are Killing vector fields for the quotient hyper-Kähler metric and generate the T^q-action induced by (19).

On $(\mu^\theta)^{-1}(\xi)$ one has

$$\operatorname{Im} X_\alpha + \xi_\alpha = \frac{1}{2}\sum_{\beta=1}^{q} \theta^\alpha_\beta \mathbf{r}_\beta, \quad \alpha = 1, \dots, l,$$

where $\xi_\alpha = (\xi_1)_\alpha i + (\xi_2)_\alpha j + (\xi_3)_\alpha k$, so the metric on $(\mu^\theta)^{-1}(0)$ is given by

$$\sum_{\alpha=1}^{p} \mathrm{d}x_\alpha^2 + \sum_{\alpha=l+1}^{p} \mathrm{d}(\operatorname{Im} X_\alpha)^2 + \frac{1}{4}\sum_{\alpha=1}^{l}\left(\sum_{\beta=1}^{q}\theta^\alpha_\beta\,\mathrm{d}\mathbf{r}_\beta\right)^2 + \frac{1}{4}\sum_{\beta=1}^{q}\left(\frac{1}{r_\beta}\mathrm{d}\mathbf{r}_\beta^2 + r_\beta(\mathrm{d}\psi_\beta + \mathbf{\Omega}_\beta\cdot\mathrm{d}\mathbf{r}_\beta)^2\right).$$

Projecting orthogonally onto the space spanned by the Killing vector fields $\frac{\partial}{\partial x_\alpha}, \alpha = 1, \dots, l$, one gets that, locally, the metric on the quotient is given by $h = h_0 + h_1$, where

$$h_0 = \sum_{\alpha=l+1}^{p} (\mathrm{d}x_\alpha^2 + \mathrm{d}(\operatorname{Im} X_\alpha)^2)$$

and h_1 is given by (23) with the matrix H as in (24). □

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